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Comment on 'Detecting non-Abelian geometric phases with three-level Λ systems'

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In their recent paper, Yan-Xiong Du *et al.* [Phys. Rev. A **84**, 034103 (2011)] claim to have found a non-Abelian adiabatic geometric phase associated with the energy eigenstates of a large-detuned Λ three-level system. They further propose a test to detect the non-commutative feature of this geometric phase. On the contrary, we show that the non-Abelian geometric phase picked up by the energy eigenstates of a Λ system is trivial in the adiabatic approximation, while, in the exact treatment of the time evolution, this phase is very small and cannot be separated from the non-Abelian dynamical phase acquired along the path in parameter space.

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In a recent paper, Yan-Xiong Du *et al.* [1] claim to have found a measurable non-Abelian geometric phase (GP) in adiabatic evolution of a Λ three-level system. At first sight, this is a somewhat paradoxical claim since a nontrivial non-Abelian GP in adiabatic evolution requires degenerate energy eigenstates [2], such as the two dark states of a tripod system [3], while the energy eigenstates of a Λ system are all non-degenerate. Therefore, the idea of Ref. [1] is to look at the Λ system in the large detuning regime, where two of the energy eigenstates become nearly degenerate and may therefore pick up a non-Abelian adiabatic GP.

Contrary to the claim in Ref. [1], we show here that the non-Abelian GP of a large-detuned Λ system in the adiabatic approximation is trivial. We further show that the non-Abelian GP picked up by this pair of nearly degenerate energy eigenstates in the exact treatment of the time evolution is small and cannot be separated from the non-Abelian dynamical phase acquired along the path in parameter space. This latter result implies that the non-commutative feature of this GP cannot be detected in an experiment.

Following Ref. [1], the considered system is a cold atomic gas, each atom having an internal three-level Λ-type configuration. The two ground state levels $|1\rangle$ and $|2\rangle$ are coupled to an excited state $|3\rangle$ by two laser beams associated with Rabi frequencies $\Omega_1 = \Omega \sin\theta e^{i\varphi}$ and $\Omega_2 = \Omega\cos\theta$. Here, θ, φ, and Ω are slowly varying parameters in time. The Hamiltonian in a frame that rotates with the laser fields reads

$$H = -\hbar\left(\Omega\sin\theta e^{i\varphi}|1\rangle\langle 3| + \Omega\cos\theta|2\rangle\langle 3| \right.$$
$$\left. +2\Delta|3\rangle\langle 3|\right) + \text{H.c.}, \qquad (1)$$

where rapidly oscillating counter-rotating terms have been neglected (rotating wave approximation). The two laser fields are assumed to have the same detuning Δ. Diagonalizing H yields the exact eigenvectors

$$\begin{aligned}
|\psi_1\rangle &= \cos\theta|1\rangle - \sin\theta e^{-i\varphi}|2\rangle, \\
|\psi_2\rangle &= \cos\gamma\left(\sin\theta e^{i\varphi}|1\rangle + \cos\theta|2\rangle\right) - \sin\gamma|3\rangle, \\
|\psi_3\rangle &= \sin\gamma\left(\sin\theta e^{i\varphi}|1\rangle + \cos\theta|2\rangle\right) + \cos\gamma|3\rangle, \qquad (2)
\end{aligned}$$

where $\tan\gamma = \left(\sqrt{\Delta^2+\Omega^2} - \Delta\right)/\Omega$. The corresponding exact energy eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -\hbar\left(\Delta - \sqrt{\Delta^2+\Omega^2}\right)$, and $\lambda_3 = -\hbar\left(\Delta + \sqrt{\Delta^2+\Omega^2}\right)$.

In the limit of large positive detuning $\Delta \gg |\Omega|$, the first two eigenstates become nearly degenerate. Furthermore, $\sin\gamma \approx 0$ in this limit, which implies that the second and third eigenvectors can be approximated as $|\psi_2\rangle \approx \sin\theta e^{i\varphi}|1\rangle + \cos\theta|2\rangle$ and $|\psi_3\rangle \approx |3\rangle$. Thus, $|\psi_3\rangle$ is decoupled from the nearly degenerate energy eigenstates $|\psi_1\rangle, |\psi_2\rangle$. These two latter states span a two-dimensional subspace $\mathcal{S}$ of the three-level system. However, $\mathcal{S}$ is independent of the adiabatic parameters θ, φ [4] and the corresponding matrix-valued Wilczek-Zee vector potential $A_{ab\mu} = \langle\psi_a|\partial_\mu|\psi_b\rangle$ [2] must therefore be a pure gauge. In other words, the non-Abelian gauge field $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu - [A_\mu, A_\nu]$ must vanish. This can be verified by explicit calculation. We first find

$$\begin{aligned}
A_\theta &\approx i\sigma_y\cos\varphi + i\sigma_x\sin\varphi, \\
A_\varphi &\approx -i\sigma_z\sin^2\theta + i\sigma_x\sin\theta\cos\theta\cos\varphi \\
&\quad -i\sigma_y\sin\theta\cos\theta\sin\varphi, \qquad (3)
\end{aligned}$$

where $\sigma_x, \sigma_y, \sigma_z$ are standard Pauli matrices defined in the $|1\rangle, |2\rangle$ basis. Notice the sign in the first term of the right-hand side in the expression for A_θ, which is opposite to that of the corresponding expression in Eq. (7) of Ref. [1]. By using Eq. (3), one verifies that $F_{\theta\varphi} \approx 0$. On the other hand, by using Eq. (7) of Ref. [1], one obtains a non-vanishing $F_{\theta\varphi}$, which contradicts the fact that $\mathcal{S}$ is independent of the adiabatic parameters θ, φ. The origin of these conflicting results regarding the gauge field $F_{\theta\varphi}$ is exactly the sign difference in the σ_y term of A_θ noted above.

Since the gauge field vanishes, it follows that the non-Abelian GP is trivial, contrary to the claim in Ref. [1]. To verify this explicitly, we need to choose a single-valued basis $|\eta_a\rangle$ of the two-dimensional subspace $\mathcal{S}$, in terms of which the vector potential $\mathcal{A}_{ab\mu} = \langle\eta_a|\,\partial_\mu\,|\eta_b\rangle$ and the non-Abelian GP $U_g = \mathbf{P}e^{-\oint \mathcal{A}_\mu dx^\mu}$ can be calculated. Two different single-valued bases $|\eta_a\rangle$ and $|\eta'_a\rangle$ can be related by a single-valued unitary 2×2 matrix V as $|\eta'_a\rangle = \sum_b |\eta_b\rangle V_{ba}$. The change $|\eta_a\rangle \mapsto |\eta'_a\rangle$ induced by V is a gauge transformation. The corresponding non-Abelian GPs U_g and U'_g are related as $U'_g = V_0 U_g V_0^\dagger$, where V_0 generates the transformation between the initial bases [5]. Now, by choosing $|\eta_a\rangle = |a\rangle$, we obtain $\mathcal{A}_{ab\mu} = \langle a|\,\partial_\mu\,|b\rangle = 0$, which implies $U_g = \hat{1}$ and $U'_g = V_0\hat{1}V_0^\dagger = \hat{1}$, where $\hat{1}$ is the 2×2 identity matrix and we have used that V_0 is unitary. This holds for any single-valued basis; in particular, if we choose $|\eta'_a\rangle = |\psi_a\rangle$, then $V = \hat{1}\cos\theta + i\sigma_y\sin\theta\cos\varphi + i\sigma_x\sin\theta\sin\varphi$ and $\mathcal{A}_\mu$ becomes identical to A_μ in Eq. (3). This demonstrates that any closed path integral of a matrix-valued gauge potential for this system, such as A_μ in Eq. (3), gives rise to a trivial non-Abelian geometric phase.

One may wonder whether the vanishing gauge field is an artefact of the neglect of $-\sin\gamma\,|3\rangle$ in $|\psi_2\rangle$ in the large detuning limit. Indeed, by including this term in the calculation, the gauge field $F_{\theta\varphi}$ turns out to be non-vanishing, and this may potentially cause a measurable non-Abelian GP effect. However, as we show next, this gauge field is very small and the effect of the non-Abelian GP can therefore be neglected in the large-detuning limit. Furthermore, it turns out that the dynamical 2×2 matrix $D_{kl} = \langle\psi_k|\,H\,|\psi_l\rangle$, $k,l = 1,2$, does not commute with the Wilczek-Zee vector potential, which implies that the non-Abelian dynamical phase $\mathbf{T}e^{-(i/\hbar)\int_0^\tau Ddt}$ cannot be canceled. In other words, the non-commutative feature of the non-Abelian GP of the energy eigenstates in the Λ system cannot be detected in an experiment.

First, we prove that the effect of the non-Abelian GP is negligible in the large detuning limit. By using the exact expressions for $|\psi_1\rangle, |\psi_2\rangle$ in Eq. (2), we obtain the Wilczek-Zee potential

$$\begin{aligned} A_\theta &= (i\sigma_y\cos\varphi + i\sigma_x\sin\varphi)\cos\gamma, \\ A_\varphi &= -i\sigma_z\sin^2\theta - i\frac{1}{2}(\hat{1}-\sigma_z)\sin^2\theta\sin^2\gamma \\ &\quad + i\sigma_x\sin\theta\cos\theta\cos\varphi\cos\gamma \\ &\quad - i\sigma_y\sin\theta\cos\theta\sin\varphi\cos\gamma, \end{aligned} \tag{4}$$

and corresponding gauge field

$$\begin{aligned} F_{\theta\varphi} = i\sin^2\gamma\Big[&\frac{1}{2}(\hat{1}+\sigma_z)\sin\theta\cos\theta \\ &+ \sigma_x\sin^2\theta\cos\varphi\cos\gamma \\ &- \sigma_y\sin^2\theta\sin\varphi\cos\gamma\Big]. \end{aligned} \tag{5}$$

Thus, the gauge field is almost vanishing since $\sin\gamma$ is small for large detuning. The associated non-Abelian GP is therefore close to the identity in this limit.

Secondly, we prove that the non-Abelian GP cannot be detected since it does not separate from the dynamical phase. To do this, we first note that the gauge potential in Eq. (4) is exact and can therefore be used to calculate formally the time evolution operator $U(\tau,0)$ acting on the subspace $\mathcal{S}'$ spanned by the exact eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$ in Eq. (2), no matter the value of Δ. For cyclic evolution of $\mathcal{S}'$, one obtains [6]

$$U(\tau,0) = \mathbf{T}e^{-\int_0^\tau(A_\theta\dot{\theta}+A_\varphi\dot{\varphi}+\frac{i}{\hbar}D)dt}, \tag{6}$$

where $\mathbf{T}$ is time ordering. Here, the dynamical matrix reads

$$D = \frac{1}{2}\hbar\Omega\tan\gamma\left(\hat{1}+\sigma_z\right), \tag{7}$$

and A_θ, A_φ are given by Eq. (4). One sees that the commutator $[D, A_\theta\dot{\theta} + A_\varphi\dot{\varphi}]$ is in the order of $\sin\gamma$, which means that the dynamical and geometric contributions only separate when $\sin\gamma = 0$. However, in this case the gauge field in Eq. (5) strictly vanishes and the non-Abelian GP as well as the dynamical phase are both trivial [8]. Thus, contrary to the claim in Ref. [1], a nontrivial non-Abelian GP $\mathbf{T}e^{-\int_0^\tau(A_\theta\dot{\theta}+A_\varphi\dot{\varphi})dt}$ is not detectable for a path in the space of slow parameters θ,φ since it cannot be separated from the dynamical part $\mathbf{T}e^{-(i/\hbar)\int_0^\tau Ddt}$.

It is instructive to compare the above with the non-Abelian GP proposed in [7] for a zero-detuned (i.e., $\tan\gamma = 1$) Λ system. In contrast to Ref. [1], this GP arises in non-adiabatic evolution, as generated by keeping the parameters θ and φ fixed, while Ω is turned on and off so that the subspace spanned by $|1\rangle$ and $|2\rangle$ performs a cyclic evolution. The resulting unitary evolution becomes purely geometric since H vanishes on this subspace for all times t. It can be proved [7] that this setting allows for non-commuting GPs that can be used to perform universal quantum computation by purely geometric means.

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[4] To see this, note that the projection operator P onto $\mathcal{S}$ reads $P = |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = |1\rangle\langle1| + |2\rangle\langle2|$, which is independent of θ, φ.

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[8] One may note that $\sin\gamma = 0$ corresponds the Hamiltonian $H = 2\Delta \left|3\right\rangle \left\langle 3\right|$, which acts trivially on $\mathcal{S}$.